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CONTROLLED SEARCH OF A MOVING OBJECT*

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Certain problems of motion control, called problems of seeking a moving object, are stated and solved. A control of the first object's motion is determined, under which a second controlled object is found whose motion the first object does not know. Conditions guaranteeing a successful completion of the search are established. Similar problems in differential games with mixed strategies were examined, for example, in /1—3/; a guaranteeing approach is used in the present paper.

1. We consider the motion of two controlled objects X and Y, described by the equations and initial conditions

$$X: x^{\cdot} = u, \ x(t_0) = x^{\circ}; \ Y: y^{\cdot} = v, \ y(t_0) = y^{\circ} \tag{1.1}$$

Here x and y are the objects' n-dimensional phase vectors, u and v are their velocities, the dot denotes derivatives with respect to time t, and t_0, x° and y° are the initial data. Objects X and Y can choose their own velocities $u(t)$ and $v(t)$ when $t \geqslant t_0$ so as to satisfy the following constraints: a) the functions $u(t)$ and $v(t)$ are piecewise-continuous for $t \geqslant t_0$; b) the inclusions

$$u(t) \in Q_x(x(t), t), \ v(t) \in Q_y(y(t), t) \tag{1.2}$$

reflecting the structure of the right hand sides of Eqs. (1.1) and the constraints on the objects' controls are fulfilled for all $t \geqslant t_0$; c) the objects' motions satisfy the constraints

$$x(t) \in D_x, \quad y(t) \in D_y \tag{1.3}$$

for $t \geqslant t_0$. Here $Q_x(x, t)$ and $Q_y(y, t)$ are prescribed closed sets in an n-dimensional space, which may depend on x and t. The initial data in (1.1) satisfy the conditions $x^\circ \in D_x$ and $y^\circ \in D_y$. When actual problems are being considered Q_x and Q_y are taken as spheres with center at the origin, while sets D_x and D_y coincide. Then constraints (1.2) become

$$|u(t)| \leqslant U, \ |v(t)| \leqslant V \tag{1.4}$$

where U and V are the maximum possible equal constant velocities of objects X and Y. Constraints (1.3) become

$$x(t) \in D, \ y(t) \in D, \ t \geqslant t_0 \tag{1.5}$$

where D is a prescribed closed set in n-dimensional space, in which the two objects can move.

Controls $u(t)$ and $v(t)$ satisfying the conditions a) – c) are called admissible. An admissible piecewise-smooth trajectory $x(t)$ or $y(t)$ corresponds to each admissible control $u(t)$ or $v(t)$. We assume that X can observe Y at instant t if and only if the observation condition

$$\{x(t), y(t)\} \in M \tag{1.6}$$

where M is a prescribed set in a $2n$-dimensional space. We present two examples of condition (1.6), reflecting real limitations on the possibility of observation.

a) Let observation be possible only if the objects are within a specified distance l from each other. Then condition (1.6) is

$$|x(t) - y(t)| \leqslant l \tag{1.7}$$

b) Let a set $E(t)$, impermeable to observation, be specified in the n-dimensional phase space; the set (a collection of barriers, possibly mobile or of changing form) may be time dependent. Observation is possible only under direct sight, i.e., when the interior of the segment XY joining points $x(t)$ and $y(t)$ has no points in common with barrier E. Condition (1.6) becomes

$$(X(t)\, Y(t)) \cap E(t) = 0 \tag{1.8}$$

(0 is the empty set).

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We pose the problem of X seeking Y.

Problem 1. Find an initial vector $x^\circ \in D_x$, a number $T \geqslant t_0$, and an admissible control $u(t)$ of object X on interval $[t_0, T]$, for which the fulfilment of condition (1.6) at some instant $t \in [t_0, T]$ is guaranteed under any initial vector $y^\circ \in D_y$ and any admissible control $v(t)$ of object Y on $[t_0, T]$.

We note that X must choose its control $u(t)$ in the form of a program, relying only on knowledge of domains Q_x, Q_y, D_x, D_y and M from (1.2), (1.3) and (1.6), having no information either on Y's control $v(t)$ or on Y's initial or current state. Obviously, the control that solves this problem will ensure the determination of any number (finite or infinite) of objects Y differing in initial data y° and admissible controls $v(t)$. Problem 1 is one of guaranteed search; similar problems (e.g., "the princess and the monster" games; see /1/) were analyzed within the framework of mixed strategies /2,3/. As a rule Problem 1 either has no solution or has an infinite set of solutions. To pick out a single solution it is natural to impose further the requirement that some functional, search time, for instance, be optimal.

Problem 2. Find an initial vector $x^0 \in D_x$ and an admissible control $u(t)$ under which Problem 1 has a solution with smallest possible T.

A number of typical concrete search problems have been solved below.

2. We consider a search problem in a plane ($n = 2$) under constraints (1.4) and (1.5) and search termination condition (1.7). The domain D in (1.5) is assumed bounded, closed and convex; its boundary is denoted Γ. At first we describe the control method being proposed and next, we show the conditions on the parameters entering into it, under which Problem 1 can be solved. Among all the directions of motion in the plane we find that one onto which the projection of domain D has minimal length. We choose a Cartesian coordinate system Ox_1x_2 such that the axis Ox_1 is along the direction mentioned and domain D lies in the strip $0 \leqslant x_1 \leqslant d$, where d is the length of the minimal projection of domain D. By construction each of the straight lines $x_1 = 0$ and $x_1 = d$ contains at least one point of boundary Γ. The points of Γ, lying on the straight lines $x_1 = 0$ and $x_1 = d$, form segments Γ_0 and Γ_1, respectively, (possibly, of zero length). Let us prove that a straight line $x_2 = \text{const}$ exists, intersecting both segments Γ_0 and Γ_1. If it did not we could find a straight line $x_2 = \text{const}$ such that segments Γ_0 and Γ_1 lay on different sides of it. But then we can turn domain D around some point of this straight line so that the whole domain is found to be strictly within the strip $0 \leqslant x_1 \leqslant d$; but this contradicts the fact that d is the minimal projection of domain D.

As axis Ox_2 we select a straight line $x_2 = \text{const}$ intersecting both Γ_0 and Γ_1; then $O \in \Gamma$ (Fig.1). Such a choice of coordinate system implies a rotation and a translation and does not change relations (1.1), (1.4) and (1.7); therefore, there is no loss of generality. As A_0 and A_* we take points on Γ having the largest and the smallest coordinate x_2 equalling x_2^+ and x_2^-, respectively, (the choice of these points may not be unique). Domain D can be specified by the inequalities

$$0 \leqslant f^-(x_2) \leqslant x_1 \leqslant f^+(x_2) \leqslant d, \quad x_2^- \leqslant x_2 \leqslant x_2^+ \tag{2.1}$$

where f^- and f^+ are functions continuous in the interval (x_2^-, x_2^+), specifying two branches of boundary Γ.

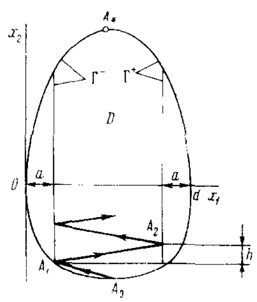

Fig.1

We take positive numbers a and h such that $a \leqslant l$ and $a \leqslant d/2$, while h is sufficiently small; these numbers are made specific below. We define curves Γ^- and Γ^+ by the relations

$$\begin{aligned} &\Gamma^\pm\colon\ x_1 = f^\pm(x_2) \pm F(x_2), \quad x_2^- < x_2 < x_2^+ \\ &F(x_2) = \max\{0, a + [f^+(x_2) - f^-(x_2) - d]/2\} \end{aligned} \tag{2.2}$$

Curves Γ^- and Γ^+ lie in domain D and are distant by no more than a from the corresponding branches (2.1) or boundary Γ. The distance between Γ^- and Γ^+ along the x_1-axis lies between the limits $[0, d - 2a]$. We construct a polynomial line $A_0A_1 \ldots A_N$, where $A_N = A_*$, the odd vertices $A_1, A_3, \ldots$ lie on Γ^- and the even vertices $A_2, A_4, \ldots$ lie on Γ^+. The coordinates x_2 of points A_i increase with i by the amount h, and by not more than h when passing from A_{N-1} to A_N. This polygonal line describes the path of point X; the magnitude of the motion's velocity is specified as maximum: $|u(t)| = U$. Thus, point X scans domain D with a step equal to h along the x_2-axis, leaving out fields of width $F(x_2) \leqslant a$ on each side of the domain (Fig.1). The fields' width equals a only where $f^- = 0$ and $f^+ = d$, as follows from (2.1) and (2.2). In particular, it equals a when $x_2 = 0$ because we chose the coordinate system such that $f^-(0) = 0$ and $f^+(0) = d$.

3. We pass to the determination of parameters a and h; for this we first consider the case of a rectangular domain D, in which case $f^- \equiv 0$ and $f^+ \equiv d$ in (2.1). We set $h = 0$ as well and we consider the motion of point X along the segment $[a, d - 2a]$ of the x_1-axis, with velocity U, where the velocity's direction reverses at the segment's endpoints. Let us ascertain the conditions under which object Y can intersect the x_1-axis, avoiding observation, i.e., staying at a distance greater than l from X. We see that Y most easily intersects the x_1-axis unnoticed by moving along boundary Γ, since then it will have available the longest time, equal to $-2(d-2a)/U$, before the next return of X on the segment's boundary. Thus, at the initial instant $t=0$ let Y occupy the position $y_1=0$, $y_2<-(l^2-a^2)^{1/2}$ outside the l-neighborhood of X and, moving along the y_2-axis, suppose that it must reach the point $y_1=0$, $y_2>(l^2-a^2)^{1/2}$ in time $2(d-(2a)/U$, remaining outside neighborhood (1.7) at all times. It is obvious that is sufficies to construct Y's motion on the time interval $[0,(d-2a)/U]$, that allows it to avoid observation and to reach point O. Then by symmetry we can construct the second half of the motion as the odd function $y_2[t-(d-2a)/U]$. Since when $t\in[0,(d-2a)/U]$ object X moves by the rule $x_1=a+Ut$, the boundary of its l-neighborhood moves on the x_2-axis by the law

$$x_2=g(t)\equiv-[l^2-(a+Ut)^2]^{1/2},\quad t\in[0,t'],\quad t'=\frac{l-a}{U} \tag{3.1}$$

The derivative $g^{\cdot}(t)$ of function (3.1) grows monotonically from $u_0=aU(l^2-a^2)^{-1/2}$ to ∞ on interval $[0,t']$ and takes the value V when

$$t=t''=VU^{-1}(U^2+V^2)^{-1/2}l-aU^{-1}<t' \tag{3.2}$$

Y's motion along the y_2-axis must satisfy the inequalities $y_2(t)<g(t)$ and $y_2^{\cdot}(t)\leqslant V$ and must reach $y_2=0$ in time t_*.

We obtain the lower bound t_* by solving a time-optimal problem under the constraints indicated. To do this we examine all possibilities. If $u_0<V$ (i.e., $t''>0$ in (3.2)), then for $t<t''$ object Y moves by the law $y_2=g(t)$ from (3.1), with a velocity less than V, and for $t\in[t'',t_*]$ with maximum velocity V. If, however, $u_0\geqslant V$ and $t''\leqslant 0$, then Y moves with velocity V for $t\in[0,t_*]$. As a result we obtain

$$t_*=t''-g(t'')V^{-1}\quad(u_0<V,\ t''>0)$$
$$t_*=-g(0)V^{-1}\quad(u_0\geqslant V,\ t''\leqslant 0) \tag{3.3}$$

Object Y can avoid observation under the condition $t_*<(d-2a)U^{-1}$. With due regard to (3.1)--(3.3) this inequality reduces to

$$\varphi(w,k)=(Ut_*+2a)l^{-1}<dl^{-1},\ w=VU^{-1},\ k=al^{-1} \tag{3.4}$$
$$\varphi(w,k)=(1+w^2)^{1/2}w^{-1}+k,\ k\leqslant k_0=w(1+w^2)^{-1/2}$$
$$\varphi(w,k)=(1-k^2)^{1/2}w^{-1}+2k,\ k\geqslant k_0$$

Object X can choose parameter a (or k in (3.4)) so as to maximize $\varphi(w,k)$ over $k\in[0,1]$. This narrows down the ranges of U and V under which Y can intersect the x_1-axis, avoiding observation. The required maximum is achieved at a single point k_* and equals

$$\varphi_*=\max_{0\leqslant k\leqslant 1}\varphi(w,k)=\varphi(w,k_*)=(1+4w^2)^{1/2}w^{-1} \tag{3.5}$$
$$k_*=2w(1+4w^2)^{-1/2},\ k_0\leqslant k_*\leqslant 1$$

From (3.4) and (3.5) it follows that if the inequality $\varphi_*<dl^{-1}$, is fulfilled, then Y, moving in the manner indicated, can intersect the x_1-axis avoiding observation. Under the reverse inequality $\varphi_*>dl^{-1}$ which by (3.5) is

$$(1+4w^2)^{1/2}w^{-1}>dl^{-1} \tag{3.6}$$

X can observe Y is the latter intersects the x_1-axis. For this, according to (3.4) and (3.5), the quantity a must be chosen as

$$a=k_*l=2w(1+4w^2)^{-1/2}l<l,\ w=VU^{-1} \tag{3.7}$$

Condition (3.6) can be solved with respect to $w=VU^{-1}$

$$(V/U)^2[(d/l)^2-4]<1 \tag{3.8}$$

Inequality (3.8) is fulfilled automatically if the simpler and cruder condition

$$V/U<l/d \tag{3.9}$$

is fulfilled.

4. We return to the general case of a convex closed bounded domain D and assume that condition (3.8) is satisfied. Let the search be conducted as described in section 2 and let a be chosen in accord with (3.7), while h is sufficiently small. At each scanning step the situation is similar to that which obtained when $h=0$, where the fields' width nowhere exceeds a, which only restricts the possibilities for object Y. Therefore, obviously, when h is sufficiently small Y cannot be found on the same straight line $x_2=\text{const}$ with X without being noticed by it. Consequently, the search is successfully completed and inequality (3.8) (and, all the more, (3.9)) is a sufficient condition for a successful completion of the search. The search time T depends on h and equals the length of the polygonal line $A_0A_1\ldots A_N$, divided by U.

Let us consider the limiting cases of inequality (3.8). If $d \leqslant 2l$, then (3.8) is fulfilled for any $V, U > 0$. In this case the search is scan-free. Object X can move with an arbitrarily small velocity U so that both branches (2.1) of boundary Γ are at distances no greater than l from it, for example, along the curve $x_1 = [f^-(x_2) + f^+(x_2)] / 2$. In the other limiting case $l \ll d$ condition (3.8) takes form (3.9). Here for a successful search X must have greater superiority in velocity. Then, according to (3.7), $al^{-1} \to 0$, so that the "fields" are practically not there.

Let us analyze the search method, somewhat different from the one in Sect.2, shown in Fig.2. Object X starts to move at point A_0 along Γ to the left up to point B_1 with coordinate $x_2 = x_2^- + h$. Next, X moves to the right along the straight line $x_2 = x_2^- + h$ up to point $B_2 \in \Gamma$, and then from it along Γ up to point B_3 with coordinate $x_2 = x_2^- + 2h$. After this X moves along the straight line $x_2 = x_2^- + 2h$ to the left up to Γ, and so on. The motion takes place at velocity U and ends at point A_*. We go on to determine the conditions and the values of $h < l$ under which this search method solves Problem 1.

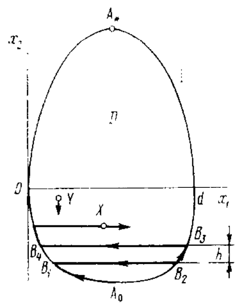

Fig.2

To avoid observation Y must intersect one of the segments B_iB_{i+1} along which X moves, in such a way that the inequality $XY > l$ is observed at all times. The most advantageous intersection point for Y is close to Γ, since it is here that X spends the longest time between motions to the left and to the right while scanning. For definiteness let us consider the situation close to the left branch of the boundary. We replace the segments of Γ above point B_{i+3} and below point B_i by segments of the straight line B_iB_{i+3} forming an angle φ with the x_2-axis, $|\varphi| < \pi / 2$ (Fig.3).Because the domain D is convex, such a replacement of the boundary can only broaden the possibilities for object Y.

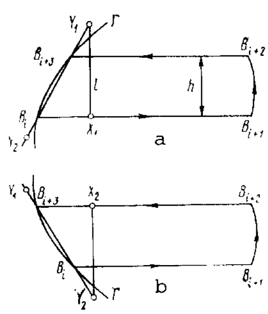

Fig.3

We first consider the case $0 \leqslant \varphi < \pi / 2$ (Fig.3a). Let Y_1 be a point on B_iB_{i+3}, located at distance l from the straight line B_iB_{i+1} and X_1 be the base of the perpendicular from Y_1 onto B_iB_{i+1}. At the instant that object X arrives at X_1 the object Y, in order not to be detected, must be located to the right of and above Y_1 on Fig.3,a. Suppose that X has travelled the path $X_1B_{i+1}B_{i+2}B_{i+3}$. Object Y, in order to avoid detection, must, at this time, be found to the left of and below a point Y_2 on B_iB_{i+3}, where $Y_2B_{i+3} = l$. Let us count the times t_1 and t_2 needed by X and Y to cover the trajectories $X_1B_{i+1}B_{i+2}B_{i+3}$ and Y_1Y_2, respectively, allowing for $B_iB_{i+1} \leqslant d$

$$t_1 \leqslant [2d + h - (l + h)\,\mathrm{tg}\varphi]\, U^{-1} \tag{4.1}$$

$$t_2 = (Y_1B_{i+3} + Y_2B_{i+3})\, V^{-1} = [(l - h)(\cos\varphi)^{-1} + l]\, V^{-1}$$

The condition for a successful search is $t_1 < t_2$, which, with due regard to (4.1), yields

$$VU^{-1} < \psi(\varphi) \equiv (l - h + l\cos\varphi)\,[(2d + h)\cos\varphi - (l + h)\sin\varphi]^{-1} \tag{4.2}$$

We can verify that $\psi'(\varphi) > 0$ when $h < l$; therefore, (4.2) is automatically fulfilled for all $\varphi \in [0, \pi / 2]$ if $VU^{-1} < \psi(0)$, i.e.

$$VU^{-1} < \psi(0) = (l - h / 2)(d + h / 2)^{-1} \tag{4.3}$$

When $\varphi \in (-\pi/2, 0)$ the situation is shown in Fig.3,b and is analyzed analogously. For a successful search the time t_1 taken by X to cover trajectory $B_iB_{i+1}B_{i+2}X_2$ must be less than the time t_2 taken by Y on Y_1Y_2. As a result we obtain the same relations (4.1) and (4.2) but with φ replaced by $-\varphi$. Therefore, (4.3) is a sufficient condition for a successful completion of the search. If (3.9) is fulfilled, then (4.3) is fulfilled for a sufficiently small h and the search method indicated is successfully completed. For this the magnitude of h must be taken from the interval

$$0 < h < 2(l - wd)(1 + w)^{-1}, \quad w = VU^{-1} < ld^{-1} \tag{4.4}$$

ensuring the fulfilment of (4.3). For the method given the search time T equals

$$T = L(D, h)\, U^{-1} = Sh^{-1}U^{-1} + O(1) \tag{4.5}$$

Here $L(D, h)$ is the length of curve $A_0B_1B_2 \ldots A_*$, a function of domain D and number h. As $h \to 0$ it asymptotically equals Sh^{-1}, where S is the area of D.

5. We turn to the search problem in a three-dimensional ($n = 3$) convex closed bounded

domain D under constraints (1.4) and (1.5) and under the observation condition (1.7).We select a Cartesian coordinate system $Ox_1x_2x_3$ such that the area of the projection of D onto the plane Ox_1x_2 is minimal. We draw the planes $x_3 = ih_0, h_0 < l, i = 0, \pm 1, \pm 2, \ldots$, and we denote the section of D by the i th plane by D_i. We prescribe X's motion as follows. In each plane $x_3 = ih_0$ forming a nonempty intersection D_i with D the point X moves as described in section 4 (Fig.2), scanning the planar domain D_i with step h. After this X passes to the next layer $x_i = (i+1) h_0$, along the boundary of domain D, and scans domain D_{i+1}, and in this way looks over all planes with nonempty D_i. The direction of scanning domains D_i changes when passing from layer to layer: from point A_0 to A_* for odd i, as in Fig.2, and from A_* to A_0 for even i. Object X passes from layer to layer along the shortest curve lying on the boundary of domain D and joining the corresponding points A_0 (or A_*) of the adjacent layers.

As the scanning parameters we select $h \in [0,2l]$ and $h_0 \in [0, l]$, starting from the requirement that the inequality $XY \leqslant l$ be fulfilled at some instant for any intersection of Y with some section D_i. For simplicity we consider a cylindrical domain D for which all the sections D_i coincide with the projection D_* of domain D onto the plane Ox_1x_2. When moving in D_i the object X approaches each point of D_i at a minimal distance no greater than $h/2$. Consequently, for Y not to be detected it must be found at a distance no less than $(l^2 - h^2/4)^{1/2}$ from the plane $x_3 = ih_0$ at some instant $t = \tau_1$. In exactly the same way, as X moves along D_{i+1} the object Y must be at the same distance from plane $x_3 = (i+1) h_0$ at some instant $t = \tau_2$ to avoid detection. Consequently, to avoid detection Y must surmount a strip of width $2(l^2 - h^2/4)^{1/2} - h_0$ in a time $\tau_2 - \tau_1$ for which the estimate

$$\tau_2 - \tau_1 < [2L(D_*, h) + h_0]\, U^{-1}$$

is valid. Here we have used formula (4.5). Therefore, if the inequality

$$[2(l^2 - h^2/4)^{1/2} - h_0]\, V^{-1} > 2[L(D_*, h) + h_0]\, U^{-1} \tag{5.1}$$

is valid, the search is successful. Condition (5.1) is fulfilled if

$$w = VU^{-1} < (l^2 - h^2/4)^{1/2} [L(D_*, h)]^{-1} \tag{5.2}$$
$$0 < h_0 < 2[(l^2 - h^2/4)^{1/2} - wL(D_* h)](1 + w)^{-1}$$

The first inequality in (5.2) contains a parameter h that is appropriately chosen to maximize the right hand side of this inequality with respect to $h \in [0,2l]$. The step h_0 is then selected in conformity with the second inequality in (5.2).

In case $l^2 \ll S$, where S is the area of D_*, formulas (5.2) simplify. From (4.5) we obtain a sufficient condition for a successful completion of the search and the optimal step $h = h_*$

$$V/U < S^{-1} \max_{0 \leqslant h \leqslant 2l} [h(l^2 - h^2/4)^{1/2}] = l^2 S^{-1}, \; h_* = l\sqrt{2} \tag{5.3}$$

By (5.2), (5.3) and (4.5) the step h_0 must be selected from the interval

$$0 < h_0 < \sqrt{2}(l - wSl^{-1})(1 + w)^{-1} \tag{5.4}$$

Relations (5.3) and (5.4) are valid when $l^2 \ll S$ for an arbitrary, not just cylindrical,domain D. Estimating the total search time by using (4.5), (5.3) and (5.4),we obtain $T \sim \Omega (h_0 h_* U)^{-1}$ when $l^2 \ll S$, where Ω is the volume of domain D.

6. Let us consider the search problem with constraints (1.4) and (1.5) under the possibility of direct sighting (1.8). Let barrier E be a convex bounded domain, with the closed domain D as its exterior. Thus, domain E is impermeable both to motion as well as to observation. In the plane case ($n = 2$) Problem 1 has a solution, obviously, if an only if $U > V$. The solution is elementary: X starts on the boundary of E and moves along it with velocity U on any side. After time $T = L(U - V)^{-1}$, where L is the length of E's boundary, X and Y are necessarily within direct sight. This solution of Problem 1 is optimal, i.e., it is as well a solution of Problem 2. When $V \geqslant U$ object Y can always move so as to be hidden behind barrier E.

In the three-dimensional case ($n = 3$) the solution of Problem 1 with conditions (1.4),(1.5) and (1.8) is substantially more complex than in the two-dimensional one. Let us construct it for a spherical domain E of radius r, impermeable to observation and to the motions of X and Y. Without loss of generality Y can be constrained to move only on the surface of sphere E. As a matter of fact, along with Y's arbitrary motion in the exterior of the sphere we consider the motion of its projection Y' onto sphere E. The velocity of projection Y' does not exceed that of Y; therefore, this motion is admissible. On the other hand, if X observes Y', it observes Y itself as well; the converse is not true. Therefore, it is more advantageous for Y to move along the surface of sphere E than outside it.

We specify X's motion as a scanning (with velocity U) of a sphere of radius $R > r$ concentric with sphere E. We set

$$\theta = \pi t T^{-1}, \; U_\theta = R\theta^{\cdot} = \pi R T^{-1} \leqslant U, \; t \in [0, T] \tag{6.1}$$
$$U_\lambda = (U^2 - U_\theta^2)^{1/2} = R\lambda^{\cdot} \sin\theta, \; \lambda(0) = 0$$

where $\theta \in [0, \pi]$ is the lattitude, λ is the longitude, and time T is chosen sufficiently large. At each instant X can observe a segment of E's surface with angular radius $\gamma = \arccos(rR^{-1})$. The center X' of the segment moves on sphere E along spiral (6.1). Object Y will be detected if in X's revolution time it is unable to intersect a loop of the spiral, having avoided observation. For a successful termination of the search it is sufficient that this condition be fulfilled at the equator where the time of revolution along a loop is maximal and equals $t_1 = 2\pi RU^{-1}$ as $T \to \infty$.

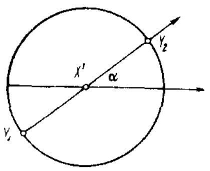

Fig.4

Let the segment's center X' move uniformly along the equator of sphere E ($T \to \infty$), accomplishing a revolution in time t_1. Object Y must intersect the equator, having avoided falling into the segment. It can be shown that the minimal t_1 in which this is possible to realize if Y moves along an arc Y_1Y_2 of a great circle intersecting the equator at an angle $\alpha = \arccos(VU^{-1})$. Figure 4 shows the segment and its center X' over equal time intervals t_1, as well as the optimal path of Y. If the time $t_2 = 2r\gamma V^{-1}$ for Y to go from Y_1 to Y_2 is less than t_1, then Y escapes observation. However, if $t_1 < t_2$, i.e., $\pi VU^{-1} < \gamma \cos\gamma$, then the search is completed successfully: $\gamma = \arccos(rR^{-1})$. Computing the maximum over $\gamma \in (0, \pi/2)$, we obtain sufficient conditions for a successful completion of the search

$$VU^{-1} < 0.179, \quad \gamma = 0.860, \quad Rr^{-1} = (\cos\gamma)^{-1} = 1.534 \tag{6.2}$$

Thus, if the first inequality in (6.2) is fulfilled, the proposed search method (6.1) solves Problem 1 when T is sufficiently large. The sphere's radius should be selected in accord with (6.2), which also yields the magnitude of the corresponding angle γ. We notice that the segment's size increases with R, but the velocity of its motion decreases; the value of R found in (6.2) is optimal for X.

In conclusion we remark that the simple search methods investigated in the paper are, in general, not optimal. The conditions obtained from them, guaranteeing successful completion of the search, are sufficient but not necessary. It would be of interest to construct optimal search methods solving Problem 2 and to obtain necessary and sufficient conditions for successful search completion.

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